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SOME REMARKS ON CONNECTED COALGEBRAS

A. ARDIZZONI AND C. MENINI

Dedicated to Freddy Van Oystaeyen, on the occasion of his sixtieth birthday

Abstract. In this paper we introduce the notions of connected, 0-connected and strictly graded coalgebra in the framework of an abelian monoidal category $\mathcal{M}$ and we investigate the relations between these concepts. We recover several results, involving these notions, which are well known in the case when $\mathcal{M}$ is the category of vector spaces over a field K. In particular we characterize when a 0-connected graded bialgebra is a bialgebra of type one.

Introduction

Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums. Given a graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta, \varepsilon)$ in $\mathcal{M}$, we can write $\Delta_{|C_n}$ as the sum of unique components $\Delta_{i,j} : C_{i+j} \to C_i \otimes C_j$ where $i + j = n$. The coalgebra C is defined to be a *strongly* $\mathbb{N}$-*graded coalgebra* (see [AM1, Definition 2.9]) when $\Delta^C_{i,j} : C_{i+j} \to C_i \otimes C_j$ is a monomorphism for every $i, j \in \mathbb{N}$. The associated graded coalgebra

$$gr_C E = C \oplus \frac{C \wedge_E C}{C} \oplus \frac{C \wedge_E C \wedge_E C}{C \wedge_E C} \oplus \cdots ,$$

for a given subcoalgebra C of a coalgebra E in $\mathcal{M}$, is an example of strongly $\mathbb{N}$-graded coalgebra (see [AM2, Theorem 2.10]).

A graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ in a cocomplete monoidal category $\mathcal{M}$ is called 0-*connected* whenever $\varepsilon_C i^C_0 : C_0 \to \mathbf{1}$ is an isomorphism where $i^C_0 : C_0 \to C$ denotes the canonical injection. C is called *strictly graded* whenever it is both strongly $\mathbb{N}$-graded and 0-connected. The associated graded coalgebra $gr_{\mathbf{1}} C$ of a coaugmented coalgebra C in $\mathcal{M}$ is an example of a strictly graded coalgebra (see Theorem 2.10). We also introduce the notion of connected coalgebra in $\mathcal{M}$ (see Definition 2.1).

In Theorem 2.11 we prove the following result. Let $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ be a 0-connected graded coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$. Then

1) $\left((C, \Delta_C, \varepsilon_C), u_C = i^C_{\mathbf{1}}\right)$ is a connected coalgebra where $u_C := i^C_0 \varepsilon_0^{-1} : \mathbf{1} \to C$;
2) $C_0 \wedge_C C_0 = C_0 \oplus P(C)$, where $P(C)$ denotes the primitive part of C.

Moreover, if $\mathcal{M}$ is also complete and satisfies $AB5$, the following assertions are equivalent:

(a) C is a strongly $\mathbb{N}$-graded coalgebra;
(b) $C_1 = P(C)$.

This result is then applied to the following setting. Let H be a braided bialgebra in a cocomplete and complete abelian braided monoidal category $(\mathcal{M}, c)$ satisfying $AB5$. Assume that the tensor product commutes with direct sums and is two-sided exact. Let M be in ${}^H_H\mathcal{M}^H_H$. Let $T = T_H(M)$ be the relative tensor algebra and let $T^c = T^c_H(M)$ be the relative cotensor coalgebra as introduced in [AMS1]. In [AM1], we proved that both T and T^c have a natural structure of graded braided bialgebra and that the natural algebra morphism from T to T^c, which coincides with the canonical injections on H and M, is a graded bialgebra homomorphism. Thus its image is a graded braided

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bialgebra which we denote by $H[M]$ and call, accordingly to [Ni], *the braided bialgebra of type one associated to* H *and* M (see [AM1, Definition 6.7]).

Let now $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ be a braided graded bialgebra in $(\mathcal{M}, c)$. Assume that B is 0-connected as a coalgebra. Then, by the foregoing, $((B, \Delta_B, \varepsilon_B), u_B)$ is a connected coalgebra. We prove (see Theorem 2.13) that B is the braided bialgebra of type one $B_0[B_1]$ associated to B_0 and B_1 if and only if

$$(\oplus_{n\geq 1} B_n)^2 = \oplus_{n\geq 2} B_n \qquad \text{and} \qquad P(B) = B_1.$$

Therefore TOBAs, as introduced in [AG, Definition 3.2.3], are exactly the braided bialgebras of type one in $\mathcal{M} = {}^H_H\mathcal{YD}$ which are 0-connected.

1. Preliminaries and Notations

Notations. Let $[(X, i_X)]$ be a subobject of an object E in an abelian category $\mathcal{M}$, where $i_X = i_X^E : X \hookrightarrow E$ is a monomorphism and $[(X, i_X)]$ is the associated equivalence class. By abuse of language, we will say that (X, i_X) is a subobject of E and we will write $(X, i_X) = (Y, i_Y)$ to mean that $(Y, i_Y) \in [(X, i_X)]$. The same convention applies to cokernels. If (X, i_X) is a subobject of E then we will write $(E/X, p_X) = \mathrm{Coker}(i_X)$, where $p_X = p_X^E : E \to E/X$.
Let $(X_1, i_{X_1}^{Y_1})$ be a subobject of Y_1 and let $(X_2, i_{X_2}^{Y_2})$ be a subobject of Y_2. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms such that $y \circ i_{X_1}^{Y_1} = i_{X_2}^{Y_2} \circ x$. Then there exists a unique morphism, which we denote by $y/x = \frac{y}{x} : Y_1/X_1 \to Y_2/X_2$, such that $\frac{y}{x} \circ p_{X_1}^{Y_1} = p_{X_2}^{Y_2} \circ y$:

$$\begin{tikzcd}
X_1 \arrow[r, hook, "i_{X_1}^{Y_1}"] \arrow[d, "x"'] & Y_1 \arrow[r, "p_{X_1}^{Y_1}"] \arrow[d, "y"'] & \frac{Y_1}{X_1} \arrow[d, "\frac{y}{x}"] \\
X_2 \arrow[r, hook, "i_{X_2}^{Y_2}"] & Y_2 \arrow[r, "p_{X_2}^{Y_2}"] & \frac{Y_2}{X_2}
\end{tikzcd}$$

$\delta_{u,v}$ will denote the Kronecker symbol for every $u, v \in \mathbb{N}$.

1.1. **Monoidal Categories.** Recall that (see [Ka, Chap. XI]) a *monoidal category* is a category $\mathcal{M}$ endowed with an object $\mathbf{1} \in \mathcal{M}$ (called *unit*), a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (called *tensor product*), and functorial isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$, $r_X : X \otimes \mathbf{1} \to X$, for every X, Y, Z in $\mathcal{M}$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom,* that is the following relation

$$(U \otimes a_{V,W,X}) \circ a_{U,V\otimes W,X} \circ (a_{U,V,W} \otimes X) = a_{U,V,W\otimes X} \circ a_{U\otimes V,W,X}$$

holds true, for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they obey the *Triangle Axiom,* that is $(V \otimes l_W) \circ a_{V,\mathbf{1},W} = r_V \otimes W$, for every V, W in $\mathcal{M}$.

A *braided monoidal category* $(\mathcal{M}, c)$ is a monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ equipped with a *braiding* c, that is a natural isomorphism $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ for every X, Y, Z in $\mathcal{M}$ satisfying

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes Y)(X \otimes c_{Y,Z}) \qquad \text{and} \qquad c_{X,Y\otimes Z} = (Y \otimes c_{X,Z})(c_{X,Y} \otimes Z).$$

For further details on these topics, we refer to [Ka, Chapter XIII].

It is well known that the Pentagon Axiom completely solves the consistency problem arising out of the possibility of going from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$ in two different ways (see [Mj1, page 420]). This allows the notation $X_1 \otimes \cdots \otimes X_n$ forgetting the brackets for any object obtained from $X_1, \cdots X_n$ using $\otimes$. Also, as a consequence of the coherence theorem, the constraints take care of themselves and can then be omitted in any computation involving morphisms in a monoidal category $\mathcal{M}$.
Thus, for sake of simplicity, from now on, we will omit the associativity constraints.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. Given an algebra A in $\mathcal{M}$ on can define the categories ${}_A\mathcal{M}$, $\mathcal{M}_A$ and ${}_A\mathcal{M}_A$ of left, right and two-sided modules over A respectively. Similarly, given a coalgebra C in $\mathcal{M}$, one can define the categories of C-comodules ${}^C\mathcal{M}, \mathcal{M}^C, {}^C\mathcal{M}^C$. For more details, the reader is referred to [AMS2].

DEFINITIONS 1.2. Let $\mathcal{M}$ be a monoidal category.
We say that $\mathcal{M}$ is a **coabelian monoidal category** if $\mathcal{M}$ is abelian and both the functors $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and left exact, for any $X \in \mathcal{M}$.

1.3. Let $\mathcal{M}$ be a coabelian monoidal category.
Let (C, i_C^E) and (D, i_D^E) be two subobjects of a coalgebra (E, Δ, ε). Set

$$\Delta_{C,D} := (p_C^E \otimes p_D^E)\Delta : E \to \frac{E}{C} \otimes \frac{E}{D}$$

$$(C \wedge_E D, i_{C\wedge_E D}^E) = \ker(\Delta_{C,D}), \qquad i_{C\wedge_E D}^E : C \wedge_E D \to E$$

$$(\frac{E}{C \wedge_E D}, p_{C\wedge_E D}^E) = \mathrm{Coker}\left(i_{C\wedge_E D}^E\right) = \mathrm{Im}(\Delta_{C,D}), \qquad p_{C\wedge_E D}^E : E \to \frac{E}{C \wedge_E D}$$

Moreover, we have the following exact sequence:

$$0 \longrightarrow C \wedge_E D \overset{i_{C\wedge_E D}^E}{\longrightarrow} E \overset{p_{C\wedge_E D}^E}{\longrightarrow} \frac{E}{C \wedge_E D} \longrightarrow 0. \tag{1}$$

Assume now that (C, i_C^E) and (D, i_D^E) are two subcoalgebras of (E, Δ, ε). Since $\Delta_{C,D} \in {}^E\mathcal{M}^E$, it is straightforward to prove that $C \wedge_E D$ is a coalgebra and that $i_{C\wedge_E D}^E$ is a coalgebra homomorphism.

Let (C, i_C^E) be a subobject of a coalgebra (E, Δ, ε) in a coabelian monoidal category $\mathcal{M}$. We can define (see [AMS2]) the n-th wedge product $\left(C^{\wedge_E n}, i_{C^{\wedge_E n}}^E\right)$ of C in E where $i_{C^{\wedge_E n}}^E : C^{\wedge_E n} \to E$. By definition, we have

$$C^{\wedge_E 0} = 0 \qquad \text{and} \qquad C^{\wedge_E n} = C^{\wedge_E n-1} \wedge_E C, \text{ for every } n \geq 1.$$

One can check that $C^{\wedge_E i} \wedge_E C^{\wedge_E j} = C^{\wedge_E i+j}$ for every $i, j \in \mathbb{N}$.

Assume now that (C, i_C^E) is a subcoalgebra of the coalgebra (E, Δ, ε). Then there is a (unique) coalgebra homomorphism

$$i_{C^{\wedge_E n}}^{C^{\wedge_E n+1}} : C^{\wedge_E n} \to C^{\wedge_E n+1}, \text{ for every } n \in \mathbb{N}.$$

such that $i_{C^{\wedge_E n+1}}^E \circ i_{C^{\wedge_E n}}^{C^{\wedge_E n+1}} = i_{C^{\wedge_E n}}^E$.

1.4. **Graded Objects.** Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of objects in a cocomplete coabelian monoidal category $\mathcal{M}$ and let

$$X = \bigoplus_{n\in\mathbb{N}} X_n$$

be their coproduct in $\mathcal{M}$. In this case we also say that X is a *graded object of* $\mathcal{M}$ and that the sequence $(X_n)_{n\in\mathbb{N}}$ defines a grading on X. A morphism

$$f : X = \bigoplus_{n\in\mathbb{N}} X_n \to Y = \bigoplus_{n\in\mathbb{N}} Y_n$$

is called a *graded homomorphism* whenever there exists a family of morphisms $(f_n : X_n \to Y_n)_{n\in\mathbb{N}}$ such that $f = \oplus_{n\in\mathbb{N}} f_n$ i.e. such that

$$f \circ i_{X_n}^X = i_{Y_n}^Y \circ f_n, \text{ for every } n \in \mathbb{N}.$$

We fix the following notations. Throughout let

$$p_n^X : X \to X_n \qquad \text{and} \qquad i_n^X : X_n \to X$$

be the canonical projection and injection respectively, for any $n \in \mathbb{N}$.

Given graded objects X, Y in $\mathcal{M}$ we set

$$(X \otimes Y)_n = \oplus_{a+b=n} (X_a \otimes Y_b).$$

Then this defines a grading on $X \otimes Y$ whenever the tensor product commutes with direct sums.

1.5. Let $\mathcal{M}$ be a coabelian monoidal category such that the tensor product commutes with direct sums.

Recall that a *graded coalgebra* in $\mathcal{M}$ is a coalgebra (C, Δ, ε) where

$$C = \oplus_{n \in \mathbb{N}} C_n$$

is a graded object of $\mathcal{M}$ such that $\Delta : C \to C \otimes C$ is a graded homomorphism i.e. there exists a family $(\Delta_n)_{n \in \mathbb{N}}$ of morphisms

$$\Delta_n^C = \Delta_n : C_n \to (C \otimes C)_n = \oplus_{a+b=n} (C_a \otimes C_b) \text{ such that } \Delta = \oplus_{n \in \mathbb{N}} \Delta_n.$$

We set

$$\Delta_{a,b}^C = \Delta_{a,b} := \left(C_{a+b} \overset{\Delta_{a+b}}{\to} (C \otimes C)_{a+b} \overset{\omega_{a,b}^{C,C}}{\to} C_a \otimes C_b \right).$$

A homomorphism $f : (C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D)$ of coalgebras is a graded coalgebra homomorphism if it is a graded homomorphism too.

DEFINITION 1.6. [AM1, Definition 2.9] Let $(C = \oplus_{n \in \mathbb{N}} C_n, \Delta, \varepsilon)$ be a graded coalgebra in $\mathcal{M}$. In analogy with the group graded case (see [NT]), we say that C is a *strongly* $\mathbb{N}$*-graded coalgebra* whenever

$$\Delta_{i,j}^C : C_{i+j} \to C_i \otimes C_j \text{ is a monomorphism for every } i, j \in \mathbb{N},$$

where $\Delta_{i,j}^C$ is the morphism defined in Definition 1.5.

2. Connected coalgebras

DEFINITIONS 2.1. Let $\mathcal{M}$ be a coabelian monoidal category. A *coaugmented coalgebra* $((C, \Delta, \varepsilon), u)$ in $\mathcal{M}$ consists of a coalgebra (C, Δ, ε) endowed with a coalgebra homomorphism $u : \mathbf{1} \to C$ called *coaugmentation* of C. Note that u is a monomorphism as $\varepsilon u = \mathrm{Id}_{\mathbf{1}}$. Given a coaugmented coalgebra $((C, \Delta, \varepsilon), u)$ define

$$\alpha_C := (C \otimes u_C) \circ r_C^{-1} + (u_C \otimes C) \circ l_C^{-1} - \Delta_C : C \to C \otimes C,$$
$$\left(P(C), i_{P(C)}\right) = \ker(\alpha_C).$$

$\left(P(C), i_{P(C)}\right)$ is called the *primitive part* of the coaugmented coalgebra C.

A *connected coalgebra* in $\mathcal{M}$ is a coaugmented coalgebra $((C, \Delta, \varepsilon), u)$ in $\mathcal{M}$ such that

$$\varinjlim (\mathbf{1}^{\wedge_C^n})_{n \in \mathbb{N}} = C.$$

REMARK 2.2. Let $\mathcal{M}$ be the category of vector spaces over a field K and let $((C, \Delta, \varepsilon), u)$ be a connected coalgebra in $\mathcal{M}$ accordingly to the previous definition. Then $C_{(0)} := Corad(C) \subseteq \mathrm{Im}(u)$ (see e.g. [AMS1, Lemma 5.2]) and hence $C_{(0)} = \mathrm{Im}(u)$ so that C is connected in the usual sense. On the other hand, since $C = \varinjlim (C_{(0)}^{\wedge_C^n})_{n \in \mathbb{N}}$ it is clear that an ordinary connected coalgebra C is also a connected coalgebra in $\mathcal{M}$.

REMARK 2.3. Let C be a connected coalgebra in the monoidal category of vector spaces over a field K. Then, the cotensor coalgebra $T^c = T_C^c(M)$ is strongly $\mathbb{N}$-graded and connected for every C-bicomodule M. Nevertheless C needs not to be K, in general.

QUESTION 2.4. Let $\mathcal{M}$ be a cocomplete coabelian monoidal category and let $((C, \Delta, \varepsilon), u)$ be a connected coalgebra in $\mathcal{M}$. Let M be a C-bicomodule in $\mathcal{M}$. Is it true that the cotensor coalgebra $T_C^c(M)$ is a connected coalgebra?

LEMMA 2.5. *Let (C, u_C) be a coaugmented coalgebra and let $f : C \to D$ be a coalgebra homomorphism in a coabelian monoidal category $\mathcal{M}$. Then (D, u_D) is a coaugmented coalgebra where $u_D = f \circ u_C$. Moreover*

$$\alpha_D \circ f = (f \otimes f) \circ \alpha_C.$$

Proof. Clearly (D,u_D) is a coaugmented coalgebra. Moreover, we have

$$\begin{aligned}
\alpha_D\circ f &= \left[(D\otimes u_D)\circ r_D^{-1}+(u_D\otimes D)\circ l_D^{-1}-\Delta_D\right]\circ f\\
&= (D\otimes f\circ u_C)\circ r_D^{-1}\circ f+(f\circ u_C\otimes D)\circ l_D^{-1}\circ f-\Delta_D\circ f\\
&= (D\otimes f\circ u_C)\circ(f\otimes\mathbf{1})\circ r_C^{-1}+(f\circ u_C\otimes D)\circ(\mathbf{1}\otimes f)\circ l_C^{-1}-(f\otimes f)\circ\Delta_C\\
&= (f\otimes f)\circ\left[(C\otimes u_C)\circ r_C^{-1}+(u_C\otimes C)\circ l_C^{-1}-\Delta_C\right]=(f\otimes f)\circ\alpha_C.
\end{aligned}$$

□

LEMMA 2.6. *Let $i_F^E:F\to E$ and $i_G^E:G\to E$ be monomorphisms which are coalgebra homomorphisms in a coabelian monoidal category $\mathcal{M}$. Then*

$$\left(\frac{F\wedge_E G}{G},{}^F\rho_{\frac{F\wedge_E G}{G}}:\frac{F\wedge_E G}{G}\to F\otimes\frac{F\wedge_E G}{G}\right)$$

is a left F-comodule where ${}^F\rho_{\frac{F\wedge_E G}{G}}$ is uniquely defined by

$${}^E\rho_{\frac{F\wedge_E G}{G}}=\left(i_F^E\otimes\frac{F\wedge_E G}{G}\right)\circ{}^F\rho_{\frac{F\wedge_E G}{G}}.$$

Furthermore the following diagram

$$\begin{tikzcd}
F\wedge_E G \arrow[rr, "\Delta_{F\wedge_E G}"] \arrow[d, "p_G^{F\wedge_E G}"'] & & (F\wedge_E G)\otimes(F\wedge_E G) \arrow[dd, "(F\wedge_E G)\otimes p_G^{F\wedge_E G}"] \\
\frac{F\wedge_E G}{G} \arrow[d, "{}^F\rho_{\frac{F\wedge_E G}{G}}"'] & & \\
F\otimes\frac{F\wedge_E G}{G} \arrow[rr, "i_F^{F\wedge_E G}\otimes\frac{F\wedge_E G}{G}"] & & (F\wedge_E G)\otimes\frac{F\wedge_E G}{G}
\end{tikzcd}$$

is commutative and

$${}^{\mathbf{1}}\rho_{\frac{\mathbf{1}\wedge_E G}{G}}=l^{-1}{}_{\frac{\mathbf{1}\wedge_E G}{G}}$$

whenever $F=\mathbf{1}$.

Proof. The first part of the statement follows by [Ar, Lemma 2.14].

Let us prove the commutativity of the diagram. We have

$$\begin{aligned}
&\left(i_{F\wedge_E G}^E\otimes\frac{F\wedge_E G}{G}\right)\circ\left(i_F^{F\wedge_E G}\otimes\frac{F\wedge_E G}{G}\right)\circ{}^F\rho_{\frac{F\wedge_E G}{G}}\circ p_G^{F\wedge_E G}\\
=&\left(i_F^E\otimes\frac{F\wedge_E G}{G}\right)\circ{}^F\rho_{\frac{F\wedge_E G}{G}}\circ p_G^{F\wedge_E G}={}^E\rho_{\frac{F\wedge_E G}{G}}\circ p_G^{F\wedge_E G}\\
=&\left(E\otimes p_G^{F\wedge_E G}\right)\circ{}^E\rho_{F\wedge_E G}=\left(i_{F\wedge_E G}^E\otimes p_G^{F\wedge_E G}\right)\circ\Delta_{F\wedge_E G}.
\end{aligned}$$

Since the tensor product is left exact, then $i_{F\wedge_E G}^E\otimes\frac{F\wedge_E G}{G}$ is a monomorphism so that we obtain the commutativity of the diagram. Finally, since

$$l^{-1}{}_{\frac{F\wedge_E G}{G}}=\left(\varepsilon_F\otimes\frac{F\wedge_E G}{G}\right)\circ{}^F\rho_{\frac{F\wedge_E G}{G}}\qquad\text{and}\qquad\varepsilon_{\mathbf{1}}=\mathrm{Id}_{\mathbf{1}},$$

when $F=\mathbf{1}$ we obtain the last equality in the statement. □

LEMMA 2.7. *Let $\left(E,u_E=i_{\mathbf{1}}^E\right)$ be a coaugmented coalgebra in a coabelian monoidal category $\mathcal{M}$. Then*

$$\left(\mathbf{1}^{\wedge_E^n},u_{\mathbf{1}^{\wedge_E^n}}=i_{\mathbf{1}}^{\mathbf{1}^{\wedge_E^n}}\right)$$

is a coaugmented coalgebra for every $n \in \mathbb{N}$. Furthermore, for every $n \in \mathbb{N}$, there exists a unique morphism $\tau_n : \mathbf{1}^{\wedge_E^{n+1}} \to \mathbf{1}^{\wedge_E^{n}} \otimes \mathbf{1}^{\wedge_E^{n}}$ such that the following diagram

$$\begin{tikzcd}
 & \mathbf{1}^{\wedge_E^{n+1}} \arrow[dl, dotted, "\tau_n"'] \arrow[d, "\alpha_{\mathbf{1}^{\wedge_E^{n+1}}}"] \\
\mathbf{1}^{\wedge_E^{n}} \otimes \mathbf{1}^{\wedge_E^{n}} \arrow[r, "i_{\mathbf{1}^{\wedge_E^{n}}}^{\mathbf{1}^{\wedge_E^{n+1}}} \otimes i_{\mathbf{1}^{\wedge_E^{n}}}^{\mathbf{1}^{\wedge_E^{n+1}}}"'] & \mathbf{1}^{\wedge_E^{n+1}} \otimes \mathbf{1}^{\wedge_E^{n+1}}
\end{tikzcd}$$

is commutative.

Proof. Set $\mathbf{1}^n := \mathbf{1}^{\wedge_E^n}$, for every $n \in \mathbb{N}$.
Since $(\mathbf{1}, u_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}})$ is a coaugmented coalgebra and $i_{\mathbf{1}^1}^{\mathbf{1}^n}$ is a coalgebra homomorphism, in view of Lemma 2.5, it is clear that $\left(\mathbf{1}^{\wedge_E^n}, u_{\mathbf{1}^{\wedge_E^n}} = i_{\mathbf{1}^1}^{\mathbf{1}^n}\right)$ is also a coaugmented coalgebra.
Consider the following exact sequence

$$0 \longrightarrow \mathbf{1}^n \xrightarrow{i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}} \mathbf{1}^{n+1} \xrightarrow{p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}} \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n} \longrightarrow 0 \tag{2}$$

were $p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}$ denotes the canonical projection. By applying the functor $\mathbf{1}^{n+1} \otimes (-)$ we get

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{1}^{n+1} \otimes \mathbf{1}^n \arrow[r, "\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}"] & \mathbf{1}^{n+1} \otimes \mathbf{1}^{n+1} \arrow[r, "\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}"] & \mathbf{1}^{n+1} \otimes \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n} \\
 & & \mathbf{1}^{n+1} \arrow[ul, dotted, "\beta_n"] \arrow[u, "\alpha_{\mathbf{1}^n}"'] &
\end{tikzcd}$$

By Lemma 2.6, we have

$$\left(i_{\mathbf{1}}^{\mathbf{1} \wedge_E \mathbf{1}^n} \otimes \frac{\mathbf{1} \wedge_E \mathbf{1}^n}{\mathbf{1}^n}\right) \circ {}^{\mathbf{1}}\rho_{\frac{\mathbf{1} \wedge_E \mathbf{1}^n}{\mathbf{1}^n}} \circ p_{\mathbf{1}^n}^{\mathbf{1} \wedge_E \mathbf{1}^n} = \left[(\mathbf{1} \wedge_E \mathbf{1}^n) \otimes p_{\mathbf{1}^n}^{\mathbf{1} \wedge_E \mathbf{1}^n}\right] \circ \Delta_{\mathbf{1} \wedge_E \mathbf{1}^n}$$

and $l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}} = {}^{\mathbf{1}}\rho_{\frac{\mathbf{1} \wedge_E G}{G}}$ so that

$$\left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\right) \circ l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}} \circ p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} = \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \Delta_{\mathbf{1}^{n+1}}. \tag{3}$$

We compute

$$\begin{aligned}
& \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \alpha_{\mathbf{1}^{n+1}} \\
= \quad & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left[\begin{array}{c} \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}} i_{\mathbf{1}^1}^{\mathbf{1}^n}\right) \circ r_{\mathbf{1}^{n+1}}^{-1} + \left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ l_{\mathbf{1}^{n+1}}^{-1} + \\ - \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \Delta_{\mathbf{1}^{n+1}} \end{array}\right] \\
= \quad & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left[\left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ l_{\mathbf{1}^{n+1}}^{-1} - \left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \Delta_{\mathbf{1}^{n+1}}\right] \\
\overset{(3)}{=} \quad & \left(i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}\right) \circ \left[\left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ l_{\mathbf{1}^{n+1}}^{-1} - \left(i_{\mathbf{1}^1}^{\mathbf{1}^{n+1}} \otimes \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\right) \circ l^{-1}{}_{\frac{\mathbf{1} \wedge_E G}{G}} \circ p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right] = 0,
\end{aligned}$$

where the last equality follows by naturality of the unit constraint.

Since $i_{\mathbf{1}^{n+1}}^E \otimes \frac{i_{\mathbf{1}^{n+1}}^E}{\mathbf{1}^n}$ is a monomorphism, we obtain

$$\left(\mathbf{1}^{n+1} \otimes p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \alpha_{\mathbf{1}^{n+1}} = 0 \tag{4}$$

so that, as the above sequence is exact, by the universal property of kernels, there exists a unique morphism $\beta_n : \mathbf{1}^{n+1} \to \mathbf{1}^{n+1} \otimes \mathbf{1}^n$ such that

$$\left(\mathbf{1}^{n+1} \otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right) \circ \beta_n = \alpha_{\mathbf{1}^{n+1}}. \tag{5}$$

By applying the functor $(-)\otimes\mathbf{1}^n$ to (2), we get

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{1}^n\otimes\mathbf{1}^n \arrow[r, "i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^n"] & \mathbf{1}^{n+1}\otimes\mathbf{1}^n \arrow[r, "p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^n"] & \frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\otimes\mathbf{1}^n \\
 & & \mathbf{1}^{n+1} \arrow[u, "\beta_n"'] \arrow[ul, dotted, "\tau_n"] &
\end{tikzcd}$$

We have

$$\begin{aligned}
\left(\frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right)\circ\left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^n\right)\circ\beta_n &= \left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^{n+1}\right)\circ\left(\mathbf{1}^{n+1}\otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right)\circ\beta_n \\
&\overset{(5)}{=} \left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^{n+1}\right)\circ\alpha_{\mathbf{1}^{n+1}} \quad = \quad 0
\end{aligned}$$

where the last equality can be proved similarly to (4). Since $\frac{\mathbf{1}^{n+1}}{\mathbf{1}^n}\otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}$ is a monomorphism we get $\left(p_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^n\right)\circ\beta_n=0$ so that, as the previous sequence is exact, by the universal property of kernels there exists a unique morphism $\tau_n:\mathbf{1}^{n+1}\to\mathbf{1}^n\otimes\mathbf{1}^n$ such that $\left(i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^n\right)\circ\tau_n=\beta_n$. Finally we have

$$\left(i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right)\circ\tau_n=\left(\mathbf{1}^{n+1}\otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right)\circ\left(i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\otimes\mathbf{1}^n\right)\circ\tau_n=\left(\mathbf{1}^{n+1}\otimes i_{\mathbf{1}^n}^{\mathbf{1}^{n+1}}\right)\circ\beta_n=\alpha_{\mathbf{1}^{n+1}}.$$

□

THEOREM 2.8. *Let $\left((E,\Delta_E,\varepsilon_E),u_E=i_{\mathbf{1}}^E\right)$ be a coaugmented coalgebra in a coabelian monoidal category $\mathcal{M}$. Then $\varepsilon_E\circ i_{P(E)}=0$ and*

$$\left(\mathbf{1}\wedge_{\mathbf{E}}\mathbf{1}=\mathbf{1}^{\wedge_E^2},i_{\mathbf{1}^{\wedge_E^2}}^E\right)=\left(\mathbf{1}\oplus P(E),\nabla\left(u_E,i_{P(E)}\right)\right),$$

where $\nabla\left(u_E,i_{P(E)}\right):\mathbf{1}\oplus P(E)\to E$ denotes the codiagonal morphism associated to u_E and $i_{P(E)}$.

Proof. Set $P=P(E)$. Since (E,u_E) is a coaugmented coalgebra, we apply Lemma 2.5 to the coalgebra homomorphism $\varepsilon_E:E\to\mathbf{1}$. Thus $\left(\mathbf{1},u_{\mathbf{1}}=\varepsilon_E i_{\mathbf{1}}^E=\mathrm{Id}_{\mathbf{1}}\right)$ is a coaugmented coalgebra and $\alpha_{\mathbf{1}}\circ\varepsilon_E=(\varepsilon_E\otimes\varepsilon_E)\circ\alpha_E$. We have

$$\alpha_{\mathbf{1}}\circ\varepsilon_E\circ i_P=(\varepsilon_E\otimes\varepsilon_E)\circ\alpha_E\circ i_P=0. \tag{6}$$

By definition, we have

$$\alpha_{\mathbf{1}}=(\mathbf{1}\otimes u_{\mathbf{1}})\circ r_{\mathbf{1}}^{-1}+(u_{\mathbf{1}}\otimes\mathbf{1})\circ l_{\mathbf{1}}^{-1}-\Delta_{\mathbf{1}}=r_{\mathbf{1}}^{-1}+l_{\mathbf{1}}^{-1}-l_{\mathbf{1}}^{-1}=r_{\mathbf{1}}^{-1}. \tag{7}$$

Since $\alpha_{\mathbf{1}}$ is an isomorphism, in view of (6), we obtain $\varepsilon_E\circ i_P=0$. Consider the following exact sequence

$$0\to\mathbf{1}\xrightarrow{i_{\mathbf{1}}^E}E\xrightarrow{p_{\mathbf{1}}^E}\frac{E}{\mathbf{1}}\to 0.$$

Since $\varepsilon_E\circ i_{\mathbf{1}}^E=\mathrm{Id}_{\mathbf{1}}$, there exists a unique morphism $a:\frac{E}{\mathbf{1}}\to E$ such that $\mathrm{Id}_E=i_{\mathbf{1}}^E\varepsilon_E+ap_{\mathbf{1}}^E$. Clearly the following sequence

$$0\to\frac{E}{\mathbf{1}}\xrightarrow{a}E\xrightarrow{\varepsilon_E}\mathbf{1}\to 0.$$

is exact. From $\varepsilon_E\circ i_P=0$, we get that there exists a unique morphism $i_P':P\to\frac{E}{\mathbf{1}}$ such that $a\circ i_P'=i_P$. Thus

$$\nabla\left(i_{\mathbf{1}}^E,a\right)\circ\left(\mathrm{Id}_{\mathbf{1}}\oplus i_P'\right)=\nabla\left(i_{\mathbf{1}}^E\circ\mathrm{Id}_{\mathbf{1}},a\circ i_P'\right)=\nabla\left(i_{\mathbf{1}}^E,i_P\right).$$

where $\nabla\left(i_{\mathbf{1}}^E,a\right):\mathbf{1}\oplus\frac{E}{\mathbf{1}}\to E$ is the codiagonal morphism associated to $i_{\mathbf{1}}^E$ and a. Since $\nabla\left(i_{\mathbf{1}}^E,a\right)$ is an isomorphism and $\mathrm{Id}_{\mathbf{1}}\oplus i_P'$ is a monomorphism, we get that $\nabla\left(i_{\mathbf{1}}^E,i_P\right)$ is a monomorphism. Let us prove that

$$\alpha_E\circ\left(i_{\mathbf{1}^{\wedge_E^2}}^E-i_{\mathbf{1}}^E\circ\varepsilon_E\circ i_{\mathbf{1}^{\wedge_E^2}}^E\right)=0.$$

By Lemma 2.5 and Lemma 2.7, we have

$$\alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \left(i^E_{\mathbf{1}^{\wedge^2_E}} \otimes i^E_{\mathbf{1}^{\wedge^2_E}}\right) \circ \alpha_{\mathbf{1}^2} = \left(i^E_{\mathbf{1}^{\wedge^2_E}} \otimes i^E_{\mathbf{1}^{\wedge^2_E}}\right) \circ \left(i^{\mathbf{1}^{\wedge^2_E}}_{\mathbf{1}} \otimes i^{\mathbf{1}^{\wedge^2_E}}_{\mathbf{1}}\right) \circ \tau_1 = \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \tau_1$$

for a suitable $\tau_1 : \mathbf{1}^{\wedge^2_E} \to \mathbf{1} \otimes \mathbf{1}$. Then, by Lemma 2.5, we have

$$(\varepsilon_E \otimes \varepsilon_E) \circ \alpha_E = \alpha_{\mathbf{1}} \circ \varepsilon_E \overset{(7)}{=} r_{\mathbf{1}}^{-1} \circ \varepsilon_E. \tag{8}$$

so that

$$\tau_1 = (\varepsilon_E \otimes \varepsilon_E) \circ \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \tau_1 = (\varepsilon_E \otimes \varepsilon_E) \circ \alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} \overset{(8)}{=} r_{\mathbf{1}}^{-1} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}.$$

Then

$$\alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \tau_1 = \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ r_{\mathbf{1}}^{-1} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}.$$

On the other, by Lemma 2.5, hand we have

$$\alpha_E \circ i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \left(i^E_{\mathbf{1}}\varepsilon_E \otimes i^E_{\mathbf{1}}\varepsilon_E\right) \circ \alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} \overset{(8)}{=} \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ r_{\mathbf{1}}^{-1} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} = \alpha_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}.$$

Hence $\alpha_E \circ \left(i^E_{\mathbf{1}^{\wedge^2_E}} - i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}\right) = 0$ so that, there exists a unique morphism $b : \mathbf{1}^{\wedge^2_E} \to P$ such that $i_P \circ b = i^E_{\mathbf{1}^{\wedge^2_E}} - i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}}$. Let

$$\Delta\left(\varepsilon_E i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) : \mathbf{1}^{\wedge^2_E} \to \mathbf{1} \oplus P$$

be the diagonal morphism of $\varepsilon_E i^E_{\mathbf{1}^{\wedge^2_E}} : \mathbf{1}^{\wedge^2_E} \to \mathbf{1}$ and $b : \mathbf{1}^{\wedge^2_E} \to P$. We have

$$\nabla\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon_E i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) = i^E_{\mathbf{1}} \circ \varepsilon_E \circ i^E_{\mathbf{1}^{\wedge^2_E}} + i_P \circ b = i^E_{\mathbf{1}^{\wedge^2_E}}$$

so that

$$\nabla\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) = i^E_{\mathbf{1}^{\wedge^2_E}}. \tag{9}$$

We have

$$\begin{aligned}
&\left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \Delta_E \circ \nabla\left(i^E_{\mathbf{1}}, i_P\right) \\
= \;& \nabla\left[\left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \Delta_E \circ i^E_{\mathbf{1}}, \left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \Delta_E \circ i_P\right] \\
= \;& \nabla\left\{\left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \left(i^E_{\mathbf{1}} \otimes i^E_{\mathbf{1}}\right) \circ \Delta_E, \left(p^E_{\mathbf{1}} \otimes p^E_{\mathbf{1}}\right) \circ \left[\left(E \otimes i^E_{\mathbf{1}}\right) \circ r_E^{-1} + \left(i^E_{\mathbf{1}} \otimes E\right) \circ l_E^{-1}\right]\right\} = 0
\end{aligned}$$

so that there exists a unique morphism $\Gamma\left(i^E_{\mathbf{1}}, i_P\right) : \mathbf{1} \oplus P \to \mathbf{1}^{\wedge^2_E}$ such that

$$\nabla\left(i^E_{\mathbf{1}}, i_P\right) = i^E_{\mathbf{1}^{\wedge^2_E}} \circ \Gamma\left(i^E_{\mathbf{1}}, i_P\right).$$

Since $\nabla\left(i^E_{\mathbf{1}}, i_P\right)$ is a monomorphism, so is $\Gamma\left(i^E_{\mathbf{1}}, i_P\right)$. On the other hand, we get

$$i^E_{\mathbf{1}^{\wedge^2_E}} \circ \Gamma\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) \overset{(9)}{=} i^E_{\mathbf{1}^{\wedge^2_E}}.$$

Since $i^E_{\mathbf{1}^{\wedge^2_E}}$ is a monomorphism, we get $\Gamma\left(i^E_{\mathbf{1}}, i_P\right) \circ \Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right) = \mathrm{Id}_{\mathbf{1}^{\wedge^2_E}}$ and hence $\Gamma\left(i^E_{\mathbf{1}}, i_P\right)$ is also an epimorphism. Thus $\Gamma\left(i^E_{\mathbf{1}}, i_P\right)$ and $\Delta\left(\varepsilon i^E_{\mathbf{1}^{\wedge^2_E}}, b\right)$ are mutual inverses. □

Definition 2.9. A graded coalgebra $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ in a cocomplete monoidal category $\mathcal{M}$ is called 0-*connected* whenever $\varepsilon_0^C = \varepsilon_C i_0^C : C_0 \to \mathbf{1}$ is an isomorphism

A graded coalgebra C in a cocomplete monoidal category $\mathcal{M}$ is called *strictly graded* whenever

1) C is 0-connected;
2) C is a strongly $\mathbb{N}$-graded coalgebra.

Next theorem provides our main example of a strictly graded coalgebra.

THEOREM 2.10. *Let $\mathcal{M}$ be a cocomplete coabelian monoidal category such that the tensor product commutes with direct sums. Let $((C,\Delta,\varepsilon),u_C)$ be a coaugmented coalgebra in $\mathcal{M}$.*

Then the associated graded coalgebra

$$gr_{\mathbf{1}}C=\mathbf{1}\oplus\frac{\mathbf{1}\wedge_C\mathbf{1}}{\mathbf{1}}\oplus\frac{\mathbf{1}\wedge_C\mathbf{1}\wedge_C\mathbf{1}}{\mathbf{1}\wedge_C\mathbf{1}}\oplus\cdots$$

is a strictly graded coalgebra.

Proof. By [AM2, Theorem 2.10], we have that $\left(gr_{\mathbf{1}}C,\Delta_{gr_{\mathbf{1}}C},\varepsilon_{gr_{\mathbf{1}}C}=\varepsilon_C\circ u_C\circ p_0^{gr_{\mathbf{1}}C}\right)$ is a strongly $\mathbb{N}$-graded coalgebra. Since u_C is a coalgebra homomorphism, we get

$$\varepsilon_{gr_{\mathbf{1}}C}=\varepsilon\circ u_C\circ p_0^{gr_{\mathbf{1}}C}=p_0^{gr_{\mathbf{1}}C}.$$

It is now clear that $\varepsilon_0^{gr_{\mathbf{1}}C}:=\varepsilon_{gr_{\mathbf{1}}C}\circ i_0^{gr_{\mathbf{1}}C}=\mathrm{Id}_{\mathbf{1}}$ so that $gr_{\mathbf{1}}C$ also 0-connected and hence it is a strictly graded coalgebra. □

THEOREM 2.11. *Let $(C=\oplus_{n\in\mathbb{N}}C_n,\Delta_C,\varepsilon_C)$ be a 0-connected graded coalgebra in a cocomplete coabelian monoidal category $\mathcal{M}$ (e.g. C is strictly graded). Then*

1) $\left((C,\Delta_C,\varepsilon_C),u_C=i_{\mathbf{1}}^C\right)$ is a connected coalgebra where $u_C:=i_0^C\varepsilon_0^{-1}:\mathbf{1}\to C$.

2) $\left(C_0^{\wedge_C^2},i^C_{C_0^{\wedge_C^2}}\right)=\left(C_0\oplus P(C),\nabla\left(i_0^C,i_{P(C)}\right)\right)$, where $\nabla\left(i_0^C,i_{P(C)}\right):C_0\oplus P(C)\to C$ denotes the diagonal morphism associated to i_0^C and $i_{P(C)}$.

Moreover if $\mathcal{M}$ is also complete and satisfies $AB5$, then the following assertions are equivalent:

(a) C is a strongly $\mathbb{N}$-graded coalgebra.

(b) $\left(C_1,i_1^C\right)=\left(P(C),i_{P(C)}\right)$.

In particular, when (b) holds, C is a strictly graded coalgebra.

Proof. 1) By Proposition [AM1, Proposition 2.5], $\left(C_0,\Delta_0=\Delta_{0,0},\varepsilon_0=\varepsilon i_0^C\right)$ is a coalgebra in $\mathcal{M}$ and i_0^C is a coalgebra homomorphism. Hence ε_0 and i_0^C are both coalgebra homomorphisms so that $\delta:=i_0^C\varepsilon_0^{-1}$ is a coalgebra homomorphism and hence $((C,\Delta_C,\varepsilon_C),\delta)$ is a coaugmented coalgebra. By [AMS1, Proposition 3.3], we have $C=\varinjlim(C_0^{\wedge_C^t})_{t\in\mathbb{N}}$. Since ε_0 is a coalgebra isomorphism, we conclude that $\varinjlim(\mathbf{1}^{\wedge_C^n})_{n\in\mathbb{N}}=C$ i.e. that $((C,\Delta_C,\varepsilon_C),\delta)$ is a connected coalgebra.

2) It follows by 1) and in view of Theorem 2.8.

Now, assume that $\mathcal{M}$ is also complete and satisfies $AB5$ and let us prove that (a) and (b) are equivalent. By 2), we have

$$\left(C_0^{\wedge_C^2},\delta_2\right)=\left(C_0\oplus P(C),\nabla\left(i_0^C,i_{P(C)}\right)\right).$$

Then, by [AM1, Theorem 2.22], (a) is equivalent to

$$\left(C_0\oplus C_1,\nabla\left(i_0^C,i_1^C\right)\right)=\left(C_0^{\wedge_C^2},\delta_2\right)$$

and hence to

$$\left(C_0\oplus C_1,\nabla\left(i_0^C,i_1^C\right)\right)=\left(C_0\oplus P(C),\nabla\left(i_0^C,i_{P(C)}\right)\right).\tag{10}$$

$(b)\Rightarrow(10)$ It is trivial.

$(10)\Rightarrow(b)$ By hypothesis there exists an isomorphism $\Lambda:C_0\oplus P(C)\to C_0\oplus C_1$ such that $\nabla\left(i_0^C,i_{P(C)}^C\right)=\nabla\left(i_0^C,i_1^C\right)\circ\Lambda$.

Let $\pi_{C_a}^{C_0\oplus C_1}:C_0\oplus C_1\to C_a$ be the canonical projection for $a=0,1$. We have

$$\varepsilon_C\circ\nabla\left(i_0^C,i_1^C\right)=\nabla\left(\varepsilon_C\circ i_0^C,\varepsilon_C\circ i_1^C\right)=\nabla\left(\varepsilon_0,0_{Hom(C_1,\mathbf{1})}\right)=\varepsilon_0\circ\pi_{C_0}^{C_0\oplus C_1}$$

and by Theorem 2.8, we have

$$\varepsilon_C\circ\nabla\left(i_0^C,i_{P(C)}\right)=\nabla\left(\varepsilon_C\circ i_0^C,\varepsilon_C\circ i_{P(C)}^C\right)=\nabla\left(\varepsilon_0,0_{Hom(P(C),\mathbf{1})}\right)=\varepsilon_0\circ\pi_{C_0}^{C_0\oplus P(C)}.$$

Hence, by definition of Λ we get

$$\varepsilon_0\circ\pi_{C_0}^{C_0\oplus C_1}\circ\Lambda=\varepsilon_C\circ\nabla\left(i_0^C,i_1^C\right)\circ\Lambda=\varepsilon_C\circ\nabla\left(i_0^C,i_{P(C)}\right)=\varepsilon_0\circ\pi_{C_0}^{C_0\oplus P(C)}.$$

Since ε_0 is an isomorphism, we get that

$$\pi_{C_0}^{C_0 \oplus C_1} \circ \Lambda = \pi_{C_0}^{C_0 \oplus P(C)}. \tag{11}$$

Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & P(C) \arrow[r, "i_{P(C)}^{C_0 \oplus P(C)}"] \arrow[d, "b"] & C_0 \oplus P(C) \arrow[r, "\pi_{C_0}^{C_0 \oplus P(C)}"] \arrow[d, "\Lambda"'] & C_0 \arrow[r] \arrow[d, "\mathrm{Id}_{C_0}"] & 0 \\ 0 \arrow[r] & C_1 \arrow[r, "i_{C_1}^{C_0 \oplus C_1}"] & C_0 \oplus C_1 \arrow[r, "\pi_{C_0}^{C_0 \oplus C_1}"] & C_0 \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact and the right square commutes. Hence there is a unique morphism $b : P(C) \to C_1$ such that the left square commutes too. Clearly b is an isomorphism. Moreover

$$i_1^C \circ b = \nabla\left(i_0^C, i_1^C\right) \circ i_{C_1}^{C_0 \oplus C_1} \circ b = \nabla\left(i_0^C, i_1^C\right) \circ \Lambda \circ i_{P(C)}^{C_0 \oplus P(C)} = \nabla\left(i_0^C, i_{P(C)}^C\right) \circ i_{P(C)}^{C_0 \oplus P(C)} = i_{P(C)}^C$$

so that $\left(C_1, i_1^C\right) = \left(P(C), i_{P(C)}^C\right)$. ☐

Remark 2.12. Let $(C = \oplus_{n\in\mathbb{N}} C_n, \Delta_C, \varepsilon_C)$ be a graded coalgebra in a cocomplete and complete coabelian monoidal category $\mathcal{M}$ satisfying $AB5$. In view of Theorem 2.11, C is strictly graded if and only if it is 0-connected and

$$\left(C_1, i_1^C\right) = \left(P(C), i_{P(C)}\right).$$

Note that, when $\mathcal{M}$ is the category of vector spaces over a field K, our definition agrees with Sweedler's one in [Sw, page 232].

Theorem 2.13. *Let $(B, m_B, u_B, \Delta_B, \varepsilon_B)$ be a braided graded bialgebra in a cocomplete and complete braided monoidal category $(\mathcal{M}, c)$ such that $\mathcal{M}$ is abelian satisfying $AB5$. Assume that the tensor products are additive, commute with direct sums and are (two-sided) exact. Assume that B is 0-connected as a coalgebra.*
Then $((B, \Delta_B, \varepsilon_B), u_B)$ is a connected coalgebra.
Moreover B is the braided bialgebra of type one $B_0[B_1]$ associated to B_0 and B_1 if and only if

$$(\oplus_{n\geq 1} B_n)^2 = \oplus_{n\geq 2} B_n \qquad \textit{and} \qquad P(B) = B_1.$$

Proof. By [AM1, Theorem 6.10] and Theorem 2.11, $((B, \Delta_B, \varepsilon_B), \delta)$ is a connected coalgebra where $\delta := i_0^B \varepsilon_0^{-1} : \mathbf{1} \to B$. Moreover B is the braided bialgebra of type one $B_0[B_1]$ associated to B_0 and B_1 if and only if $(\oplus_{n\geq 1} B_n)^2 = \oplus_{n\geq 2} B_n$ and $P(B) = B_1$.

Let us prove that $\delta = u_B$.

By [AM1, Propositions 2.5 and 3.4], $\varepsilon_B = \varepsilon_0 \circ p_0^B$ and $u_B = i_0^B \circ u_0$ where $u_0 = p_0^B \circ u_B$. Thus

$$\mathrm{Id}_{\mathbf{1}} = \varepsilon_B \circ u_B = \varepsilon_0 \circ p_0^B \circ i_0^B \circ u_0 = \varepsilon_0 \circ u_0$$

and hence $u_0 = \varepsilon_0^{-1}$. Then we get $u_B = i_0^B \circ u_0 = i_0^B \circ \varepsilon_0^{-1}$. ☐

Remark 2.14. Recall that a *TOBA* (also called braided Hopf algebra of type one) [AG, Definition 3.2.3] in the category ${}_H^H\mathcal{YD}$ of Yetter Drinfeld modules over an ordinary K-Hopf algebra H is a graded bialgebra $T = \oplus_{n\in\mathbb{N}} T_n$ in this category such that

$$T_0 \simeq K, \qquad (\oplus_{n\geq 1} T_n)^2 = \oplus_{n\geq 2} T_n, \qquad \text{and} \qquad P(T) = T_1.$$

Therefore TOBAs are exactly the bialgebras described in Theorem 2.13 in the case when $\mathcal{M} = {}_H^H\mathcal{YD}$.

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University of Ferrara, Department of Mathematics, Via Machiavelli 35, Ferrara, I-44100, Italy
E-mail address: `rdzlsn@unife.it`
URL: `http://www.unife.it/utenti/alessandro.ardizzoni`

University of Ferrara, Department of Mathematics, Via Machiavelli 35, Ferrara, I-44100, Italy
E-mail address: `men@unife.it`
URL: `http://www.unife.it/utenti/claudia.menini`